THE DIMER ENTROPY OF THE KING'S GRAPH

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Abstract. We compute the entropy per site s of the perfect matchings of the king's graph, the square lattice with both diagonals; equivalently, of the domicule tilings counted in OEIS A239264. The graph is not planar, and its boards with both sides at least 3 admit no Pfaffian orientation, so Kasteleyn's method does not apply. Row transfer matrices on cylinders of circumference up to 25, split by a conserved parity of the dimers crossing a loop, converge exponentially; a Lüscher-type extrapolation with predictive backtests and an error budget gives

$$s = 0.5655119110413253 \pm 6 \cdot 10^{-16},$$

a non-rigorous estimate. Computer-assisted Collatz–Wielandt certificates, combined with a bound for the matchings across a cut that depends on the runs of crossing cells, prove $0.545343749727 \le s \le 0.582767477484$. We also estimate the surface free energy of a free edge, $\sigma = 0.242017935766251 \pm 9 \cdot 10^{-15}$, the corner free energy, which gives the asymptotics of A239273, and the decay length $\xi = 1.235 \pm 0.004$ of the splitting between the two parity sectors. Two sign properties of the strip growth rates, convexity in the even width and a one-sided bound on the odd increments, hold numerically and would give $0.565511911035 \le s \le 0.565511911049$.

1. Introduction

A *domicule* is a pair of cells of the square grid that share a side or a corner. A domicule tiling of a $K \times L$ board is a partition of its cells into domicules. Two diagonal domicules may cross inside a unit square; for example, the 2×2 board has 3 tilings, one of which uses both diagonals. Equivalently, a domicule tiling is a perfect matching of the *king's graph* on the board, the graph whose edges join cells at chessboard king distance 1. The numbers $Z(K, L)$ of such tilings form the array A239264 of the OEIS [14], and $Z(2n, 2n)$ is A239273 [15]; related entries such as A239265, A239268 and A243510 [16, 17, 18] record further exact counts. None of these entries gives the growth constant, i.e. the entropy per site

$$s = \lim_{K,L\to\infty} \frac{\log Z(K, L)}{KL}.$$

For dominoes (the square lattice) s is known exactly, G/π with Catalan's constant G, by the Pfaffian method of Kasteleyn and of Temperley and Fisher [20, 37]. Kasteleyn's method needs a Pfaffian orientation, and planar graphs have one [23, Thm. 8.3.4]. The triangular lattice (one diagonal per square) is planar: Montroll gave an orientation for it [27], its entropy is tabulated by Wu [45], and Fendley, Moessner and Sondhi computed its correlations [7]. The king's graph is not planar (the 5×5 board already has $72 > 3 \cdot 25 - 6$ edges), but planarity is only sufficient for a Pfaffian orientation. We show that no king board with both sides at least 3 has one (Theorem 2.1), so a single Pfaffian cannot count its domicule tilings, and no determinantal formula for Z is known. Wildeboer and Seidel [44] give a "pre-Kasteleyn" orientation of this lattice under which the Pfaffian weights each dimer covering D by $(-1)^{n_c(D)}$, up to a sign that

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may depend on the topological sector, where $n_c(D)$ is the number of pairs of crossing diagonal dimers; on a board, which has a single sector, it follows that the Pfaffian is $\pm\sum_D(-1)^{n_c(D)}$. They note that the unsigned problem is not addressable by Pfaffian methods. Yao and Kivelson [47] write the variant without crossing dimers as a Grassmann integral with a quartic interaction. Wildeboer, Nussinov and Seidel [43] and Wildeboer et al. [42] solve different, constrained dimer models on lattices with crossed plaquettes.

Sandvik and Moessner [35] (see also the earlier preprint [34]) simulated dimers on the nearest- and next-nearest-neighbour bonds of the square lattice by Monte Carlo, at a small fugacity w_2 of the diagonal dimers between e^{-6} and e^{-3}; they show that the diagonal dimers deconfine monomers, with a correlation length $\propto 1/w_2$. The equal-weight point $w_2 = 1$ studied here was not simulated, and no entropy is given. Finch's errata [8, first volume, §5.12] list a constant $1.34254\ldots1.34265$ for the square lattice with "crossed diagonal bonds"; it is the growth constant $1.342643951124\ldots$ of non-attacking kings (independent sets of the king's graph, OEIS A247413 [30]), not a dimer constant. Series expansions for dimer and monomer–dimer problems on the standard two- and three-dimensional lattices go back to Nagle [28] and Gaunt [12]; Nagle's expansion has the Bethe approximation as its zeroth order (Table 6). To our knowledge no numerical value of s has been published.

Related methods. Calkin and Wilf [4] bounded the hard-square entropy by eigenvalues of strip transfer matrices; Friedland and Peled [9] developed the approach for monomer–dimer systems, and Louidor and Marcus [22] sharpened the lower bounds with Rayleigh quotients. These bounds converge to the entropy (for hard squares the differences of consecutive strip entropies converge exponentially [32]), but their proofs use a symmetric transfer matrix (in [9] a hidden symmetry), and on the king lattice the gluing form across a cut between two half-boards is indefinite (§8). Pavlov [32], Marcus and Pavlov [25] and Gamarnik and Katz [11] compute entropies rigorously from strong spatial mixing (in [32] a version implied by disagreement percolation [25]). For perfect matchings, the parity of the number of dimers crossing a closed curve is fixed by the number of sites it encloses, a non-local constraint that these arguments do not handle as they stand. Without rigour, corner transfer matrices gave the hard-square entropy to 43 decimals [2], and their renormalization-group variant [29] is a natural route to more digits here; the crossing diagonals and the parity sector of §2 need a dedicated encoding, which we leave open. We use row transfer matrices because they are exact, easy to check against brute force, and give the estimate and the certificates from the same code.

Results.

(1) No king board with both sides at least 3 has a Pfaffian orientation (Theorem 2.1).
(2) $s = 0.5655119110413253 \pm 6 \cdot 10^{-16}$ (§3), i.e. $2s = 1.13102382208265$ per dimer and $e^s = 1.76034869404198$, both rounded to 14 decimals. The error bar is not rigorous.
(3) Rigorously, $0.545343749727 \le s \le 0.582767477484$ (Theorem 4.1); the upper bound uses the run structure of the matchings across a cut (Lemma 4.6).
(4) The surface free energy of a free edge is $\sigma = 0.242017935766251 \pm 9 \cdot 10^{-15}$, and the corner free energy is $\kappa = 0.36319625605 \pm 3 \cdot 10^{-10}$, which gives the asymptotics of A239273 (§5).
(5) On even cylinders the two parity sectors split by $\Delta_w \approx e^{-w/\xi}$ with $\xi = 1.235 \pm 0.004$, and all finite-size corrections we see are exponentially small (§6), in contrast with the critical square lattice.
(6) Convexity of $q \mapsto \log \lambda_{2q}$ (L_∞) and a bound on the odd increments (O) of the strip growth rates hold wherever we can compute and would give $0.565511911035 \le s \le 0.565511911049$ (§8). The stronger log-convexity of the finite counts is not structural; a companion note [33] treats it.

The same code reproduces the exactly known entropies of the square and triangular lattices (§7). All scripts, outputs and the map from every number in this paper to the file it comes from are in the repository (§9).

2. Model, transfer matrix and parity sectors

2.1. **Definitions.** The king's graph has vertex set $\mathbb{Z}^2$, and $(r,c) \sim (r',c')$ if $\max(|r-r'|,|c-c'|) = 1$. The board $B(K,L) = \{0,\dots,K-1\} \times \{0,\dots,L-1\}$ has K rows and L columns; $Z(K,L)$ is the number of perfect matchings of the induced subgraph ($Z = 0$ if KL is odd), and $Z(K,L) = Z(L,K)$. The cylinder of circumference $w \geq 3$ identifies column c with $c+w$; its count is $Z_{\mathrm{cyl}}(K,w)$. Let

$$\bar{s} = \limsup \frac{\log Z(K,L)}{KL}, \qquad \underline{s} = \liminf \frac{\log Z(K,L)}{KL},$$

over $K,L \to \infty$ with KL even. For the dimer problem on $\mathbb{Z}^d$ the two limits are equal [9]; for the king's graph this is shown below, and Theorem 4.1 bounds $\underline{s}$ and $\bar{s}$ separately and does not use it.

For a strip of width k, stacking boards gives $Z(2i+2j,k) \geq Z(2i,k)Z(2j,k)$, so by Fekete's lemma [6] the limit

$$\lambda_k^2 = \lim_{j\to\infty} Z(2j,k)^{1/j} = \sup_j Z(2j,k)^{1/j} \tag{1}$$

exists. We call λ_k the growth rate of the free strip and write $a_k = \log \lambda_k$. Placing strips side by side gives $\lambda_{k_1+k_2} \geq \lambda_{k_1}\lambda_{k_2}$, so $a_k/k \to s^* = \sup_k a_k/k$. By (1), $Z(2j,L) \leq \lambda_L^{2j}$, so $\log Z(2j,L)/(2jL) \leq a_L/L \leq s^*$; for odd K (then L is even) $Z(K,L) \leq Z(K+1,L)$ (add a row of horizontal dominoes). Hence $\bar{s} \leq s^*$, and with $\underline{s} \geq s^*$ (the proof of the lower bound in §4) the limit s exists and equals $s^* = \lim_k a_k/k$.

2.2. **Transfer matrix.** We scan a strip of width w cell by cell in row-major order. A cell is either already covered, or it is matched to one of its later neighbours: right $(r,c+1)$, down-left $(r+1,c-1)$, down $(r+1,c)$ or down-right $(r+1,c+1)$, if that neighbour exists and is free. The state records the occupancy of the next $w+2$ cells; on the cylinder the neighbours are taken modulo w and one extra bit records the cell $(r+1,w-1)$ claimed by $(r,0)$ through the wrap-around down-left edge. A row is w cell steps, $T = T_{w-1}\cdots T_0$, and

$$Z(K,w) = e_0^{\top} T^K e_0,$$

with e_0 the empty state. Between rows only the w bits of the next row can be set, so T acts on 2^w states: the set $A \subseteq \{0,\dots,w-1\}$ of cells of the next row already covered from the current one.

2.3. **Parity sectors.** Let A_r be the set of cells of row r covered from row $r-1$ and h_r the number of dimers inside row r. Counting the cells of row r gives $w = |A_r| + 2h_r + |A_{r+1}|$, so

$$|A_{r+1}| \equiv w - |A_r| \pmod 2.$$

Hence T^2 preserves the parity p of $|A|$; for even w so does T, for odd w the map T exchanges the two sectors and T^2 has the same nonzero spectrum on both. We write $\lambda_w^{(p)}$ for the square root of the leading eigenvalue of T^2 on sector p of the cylinder, and $f_w^{(p)} = \log \lambda_w^{(p)}/w$. On the cylinder $|A|$ is the number of dimers crossing a closed loop around the axis, and for even w its parity is conserved: this is the topological $\mathbb{Z}_2$ sector familiar from dimer models on non-bipartite lattices [26, 47]. For even w the two sector values differ, and they bracket s in all our data: $f_w^{(0)} > s > f_w^{(1)}$.

For free strips only the sector of e_0 enters Z, and the spectral radius of T is not the growth rate λ_w in general: for $w = 6, 10, 14, 18$ the other sector has the larger eigenvalue (e.g. $w = 6$: $\log\lambda = 2.909568954808860$ there, against $a_6 = 2.909403518708946$). The difference of the two sectors of the free strip of even width k alternates in sign with period 4 and shrinks by a factor 13.84 per two columns ($k \le 18$).

2.4. **Computation.** We use float64 power iteration on T^2 restricted to one sector, from a random positive start vector (fixed seed), normalised every step, stopped when the estimate of $\log\lambda$ changes by less than $4\cdot10^{-15}$ for three consecutive iterations (19 iterations for $w \ge 20$). The changes of the estimate shrink by a factor 0.115 per iteration (the gap of §6), so the truncation error after stopping is far below one ulp of $\log\lambda$. Afterwards the Collatz–Wielandt bracket $\min_i (T^2v)_i/v_i \le \lambda^2 \le \max_i (T^2v)_i/v_i$ [5, 41] is evaluated. In every run v is positive on the whole sector (2^{w-1} states, `data/eig/`), so the classical bounds apply as stated. For $w = 20, \dots, 25$ the bracket closes to within one ulp of $\log\lambda$. Computed without directed rounding, it is a diagnostic here, not a certificate (the certificates are in §4.3). Widths $w \le 19$ take minutes; $w = 25$ (one sector suffices for odd w) works on vectors of 2^{28} doubles and needs 8.4 GiB and 22.5 minutes on four cores.

2.5. **No Pfaffian orientation.** An orientation of a graph is *Pfaffian* if all perfect matchings have the same sign in the Pfaffian of the oriented adjacency matrix, which then equals the number of perfect matchings up to sign [23, Ch. 8]. Planar graphs have Pfaffian orientations [23, Thm. 8.3.4], but so do some non-planar graphs [23, p. 323], so the non-planarity of the king's graph does not by itself rule out Kasteleyn's method.

Theorem 2.1. *Let KL be even. The king board $B(K, L)$ has a Pfaffian orientation if and only if* $\min(K, L) \le 2$.

Proof. If $\min(K, L) = 1$ the board is a path. The board $B(2, L)$ is planar: each unit square with its two diagonals is a K_4, consecutive squares share a vertical edge, and gluing two planar graphs along an edge that lies on the outer face of both gives a planar graph. Both have Pfaffian orientations [23, Thm. 8.3.4].

Let $\min(K, L) \ge 3$. A subgraph H of G is *conformal* if H and $G - V(H)$ both have perfect matchings. A Pfaffian orientation of G restricts to a Pfaffian orientation of every conformal H: fix a perfect matching R of $G - V(H)$; the sign of $M \cup R$ in G is the sign of M in H times a factor that depends only on R and on the vertex order, so all perfect matchings M of H have the same sign. If L is even, the corner $B(3, 4)$ (rows $0, 1, 2$, columns $0, \dots, 3$) is conformal in $B(K, L)$: the rest of the first three rows has even width $L - 4$, the other rows have even length L, and horizontal dominoes cover both. If L is odd, then K is even and $K \ge 4$, and the corner $B(4, 3)$ is conformal: horizontal dominoes cover the rest of the first four rows (width $L - 3$, even) and vertical dominoes the last $K - 4$ rows. Since $B(4, 3) \cong B(3, 4)$, it remains to show that $B(3, 4)$ has no Pfaffian orientation.

Fix the orientation D_0 of every edge from the lower to the higher vertex index, and a perfect matching M_0. Reversing the edges of a set X multiplies the sign of a perfect matching M by $(-1)^{|X\cap M|}$, so a Pfaffian orientation exists if and only if the system

$$\sum_{e\in M\triangle M_0} x_e \equiv \epsilon(M) + \epsilon(M_0) \pmod 2 \qquad \text{for all perfect matchings } M$$

has a solution $x \in \{0, 1\}^E$, where $(-1)^{\epsilon(M)}$ is the sign of M under D_0. For $B(3, 4)$, with 29 edges and 43 perfect matchings, the system is inconsistent; Gaussian elimination over GF(2) finds the contradiction after 23 of the equations. □

The elimination is done in exact integer arithmetic by `src/pfaff_rigor.py`, which also confirms that $B(2, L)$ is Pfaffian for $L \leq 12$, that $B(3, 6)$, $B(4, 4)$, $B(4, 5)$ and $B(4, 6)$ are not, and, for eleven boards up to $B(11, 12)$, that the matchings extending a fixed domino tiling of the complement of the corner already give an inconsistent subsystem.

Remark 2.2. The cylinder of circumference $w \geq 3$ with $K \geq 3$ rows (Kw even) and the corresponding torus contain $B(K, w)$ as a spanning, hence conformal, subgraph, so they have no Pfaffian orientation either. Theorem 2.1 excludes a single Pfaffian; it does not exclude other determinantal formulas, and it does not imply hardness. The perfect matchings of a graph of genus g are a linear combination of 4^g Pfaffians [10, 38]; with $V = KL$ vertices and $E = 4KL - 3K - 3L + 2$ edges, Euler's formula gives $g \geq (E - 3V + 6)/6 = ((K-3)(L-3) - 1)/6$, that is $g \geq 74$ for the 24×24 board. On a board, the signed sum of [44] is, up to sign, a single Pfaffian.

3. The value of s

3.1. **Data.** Table 1 lists the odd cylinders. The values f_w increase to s; the differences $d_w = f_w - f_{w-2}$ decay by a ratio $r_w = d_{w-2}/d_w$ that drifts from 13.4 at $w = 9$ to 10.0 at $w = 25$ (the table starts at $w = 13$). The geometric (Aitken) estimate $s_w = f_w + d_w/(r_w - 1)$ settles at 0.5655119110413253. For even w the sector values bracket s:

$w = 18$: [0.5655119060934878, 0.5655119160322841],
$w = 20$: [0.5655119102001319, 0.5655119118867463],
$w = 22$: [0.5655119108963251, 0.5655119111867462],
$w = 24$: [0.5655119110160359, 0.5655119110666571], midpoint 0.5655119110413465,

and the midpoints converge to s at the same rate as the odd widths. They come from other cylinders, but from the same code; an independent code is compared at the end of this section.

Table 1. Odd cylinders: $f_w = \log \lambda_w / w$, $d_w = f_w - f_{w-2}$, $r_w = d_{w-2}/d_w$ and the geometric extrapolation s_w, from the float64 eigenvalues; their error, at most $6.2 \cdot 10^{-17}$ per site, can move the last digit of f_w by one. At $w = 25$ the extended-precision inputs give $d_{25} = 5.991 \cdot 10^{-14}$, $r_{25} = 9.998$ and $s_{25} = 0.56551191104132523$.

w	f_w	d_w	r_w	s_w
13	0.5655119032044848	$7.943 \cdot 10^{-8}$	11.68	0.5655119106414456
15	0.5655119103137916	$7.109 \cdot 10^{-9}$	11.172	0.5655119110126792
17	0.5655119109720268	$6.582 \cdot 10^{-10}$	10.801	0.5655119110391899
19	0.5655119110345892	$6.256 \cdot 10^{-11}$	10.521	0.5655119110411600
21	0.5655119110406597	$6.071 \cdot 10^{-12}$	10.306	0.5655119110413121
23	0.5655119110412587	$5.989 \cdot 10^{-13}$	10.136	0.5655119110413242
25	0.5655119110413186	$5.994 \cdot 10^{-14}$	9.992	0.5655119110413253

3.2. **Extrapolation models and error bar.** All extrapolations below use the cylinders recomputed in x87 extended precision (64-bit mantissa): the odd widths $w = 5, \ldots, 25$ and both sectors of the even widths $w = 4, \ldots, 24$. Float64 enters only through a separate rounding term.

Main model. The corrections come from the gap inside a sector (§6): at $w = 20$ ARPACK gives $m = \log(\lambda_1/|\lambda_3|) = 1.082727$ per row, and the power iterations give the same value at every $w = 20, \ldots, 25$. In a massive quantum field theory the finite-size mass shift of a stable

particle decays exponentially, at a rate that depends on the particle and on the spectrum of light particles [24]; by analogy we fit the form

$$d_w = Ce^{-mw}w^{-3/2}\Big(1+\frac{c_1}{w}\Big),$$

whose power $w^{-3/2}$ and factor $1+c_1/w$ are our ansatz, with m fixed at the gap and C, c_1 fitted through d_{W-2}, d_W (L1), or with a further term c_2/w^2 through three differences (L2). The tail $s - f_W = \sum_{j\geq 1} d_{W+2j}$ is then summed. As checks we use the generic models of the first version of this work: geometric (G, constant ratio r_W); $d_w = Ae^{-w/\xi}w^b$ through three points (PE) or by least squares on six (PE6); $r_w = r_\infty + c/w$ (R1) and $r_w = r_\infty + c/w + e/w^2$ (R2); two exponentials by Prony's method (2E); and Wynn's ε algorithm [46] on f_w, its last full diagonal entry (WY) and the one before (WY−). The same models are applied to the even-width sector midpoints, a second sequence from other cylinders computed by the same code.

TABLE 2. Estimates of s by model on extended-precision inputs: from the odd widths up to $W = 25$ and from the even midpoints up to $W = 24$, with the backtest change of each, the difference from the estimate with the last width removed.

model	odd, $W=25$	midpoints, $W=24$	backtest change	
			odd	midpoints
L1	0.56551191104132532	0.56551191104132532	$1.4\cdot10^{-17}$	$6.5\cdot10^{-17}$
L2	0.56551191104132532	0.56551191104132532	$1.5\cdot10^{-18}$	$3.5\cdot10^{-18}$
G	0.56551191104132523	0.56551191104132564	$1.0\cdot10^{-15}$	$3.5\cdot10^{-15}$
PE	0.56551191104132532	0.56551191104132531	$1.7\cdot10^{-17}$	$8.1\cdot10^{-17}$
PE6	0.56551191104132533	0.56551191104132528	$6.7\cdot10^{-17}$	$6.0\cdot10^{-16}$
R1	0.56551191104132532	0.56551191104132531	$4.4\cdot10^{-17}$	$1.9\cdot10^{-16}$
R2	0.56551191104132532	0.56551191104132532	$6.1\cdot10^{-19}$	$1.6\cdot10^{-17}$
2E	0.56551191104132531	0.56551191104132534	$6.0\cdot10^{-17}$	$2.5\cdot10^{-16}$
WY	0.56551191104132532	0.56551191104132532	$4.0\cdot10^{-18}$	$5.4\cdot10^{-17}$
WY−	0.56551191104132532	0.56551191104132528	$7.2\cdot10^{-18}$	$9.3\cdot10^{-17}$

Predictive backtest. A model is kept if its backtest change is below 1% of the last correction $|d_W|$ of its sequence ($6.0\cdot10^{-16}$ for the odd widths, $1.9\cdot10^{-15}$ for the midpoints). Every model passes on both sequences except G, which ignores the drift of r_w (the PE fit attributes it to a power prefactor, $\xi_{\text{odd}} = 0.929$, $b = -1.80$). Measured against $|d_W|$, the backtest changes of L1 stay at $2\text{–}4\cdot10^{-5}$ on the odd widths for every cut $W = 19, 21, 23$, and those of L2 and R2 fall to $1\text{–}2\cdot10^{-6}$ at $W = 23$. The 18 kept estimates lie in $[0.56551191104132527,\ 0.56551191104132534]$.

Error budget. We add four terms:

(1) model form, $9.0\cdot10^{-17}$: the half-spread of the kept estimates ($3.0\cdot10^{-17}$) plus the largest predicted remaining error, each model's backtest change scaled by the ratio $|d_W/d_{W-2}|$ of the last two corrections ($6.0\cdot10^{-17}$);

(2) a hidden slow mode, $3.4\cdot10^{-17}$: we add a term $Bx^{(w-25)/2}$ with $x = e^{-2/1.2351}$, the slowest decay seen anywhere in the data (the parity splitting of §6), and fit it together with L1 (C, c_1 and B) by relative least squares on $w = w_0, \dots, 25$ for $w_0 = 11, 13, 15, 17$. The fitted B depends on the window, between $0.9\cdot10^{-3}\,d_{25}$ ($w_0 = 11$) and $1.95\cdot10^{-3}\,d_{25}$ ($w_0 = 13$). We allow every B within three standard errors of a fitted value, over all windows and widened to include $B = 0$: $-1.6\cdot10^{-3}\,d_{25} \leq B \leq 3.4\cdot10^{-3}\,d_{25}$. Since s is linear in B, the larger shift at the two ends is the term. The next odd width does

not show such a mode: the independent code of `replication/`, described at the end of this section, gives d_{27} within $1.5 \cdot 10^{-4}$ of the L1 prediction from $W = 25$, where a mode with $B = 1.95 \cdot 10^{-3}\, d_{25}$ would move it by $1.0 \cdot 10^{-3}$. This is consistent with the fitted B taking up the misfit of L1 at small w, though it rests on one width. Read as a mode, the whole deviation allows $|B| \le 2.8 \cdot 10^{-4}\, d_{25}$, which would move the main estimate by at most $1.4 \cdot 10^{-18}$; this check is not part of the budget;

(3) the gap, $6 \cdot 10^{-20}$: the change of L1 when m runs over the measured gaps;
(4) rounding, $1.5 \cdot 10^{-16}$: the largest change of a kept model between float64 and extended-precision inputs (the float64 values of $\log \lambda_w$ differ by at most $6.2 \cdot 10^{-17}$ per site over 33 cylinders).

The total is $2.7 \cdot 10^{-16}$. We quote a little more than twice this,

$$\boxed{s = 0.5655119110413253 \pm 6 \cdot 10^{-16}}$$

with the main model L1 on the odd widths, $s = 0.5655119110413253176$, as the centre. This is an estimate, not a bound: the budget assumes that no correction slower than those in the data appears beyond $w = 25$.

Further digits. Each further digit of the truncation error needs about $\log 10/m = 2.13$ more widths. The state vector doubles with each width, and with it the peak memory of `src/cylw.py` (4.3 GiB at $w = 24$, 8.4 GiB at $w = 25$), so a digit costs about $2^{2.13} = 4.4$ times the memory, and the run time grows by about the same factor. The quoted error bar would not follow, because its largest term is the rounding term ($1.5 \cdot 10^{-16}$ of $2.7 \cdot 10^{-16}$), which measures the float64 inputs and does not shrink with more widths. The extended-precision inputs agree with the independent code below to $7.8 \cdot 10^{-20}$ per site, about 19 digits, which is where the 64-bit mantissa of the x87 `long double` ends. Restricting the full row transfer to zero momentum and to states even under reflection would divide the state space by about $2w$, but the cell steps used here break the rotation symmetry of the intermediate states, so this needs a different implementation. Beyond a few more digits the natural tools are the corner transfer matrices and their renormalization-group variant mentioned in the introduction, with the parity sector built into the encoding.

Cross-checks of the float64 eigenvalues. Exact integer ratios $Z_{\mathrm{cyl}}(K+2, w)/Z_{\mathrm{cyl}}(K, w)$ for $w = 9, 11, 12, 13$ agree with the power iteration to $4.1 \cdot 10^{-16}$ in $\log \lambda$, and ARPACK on the full $w = 20$ transfer reproduces both sector values to $1.8 \cdot 10^{-15}$. The exact counts of the square boards give an independent, eigenvalue-free check of s, discussed with the corner free energy in §5.1.

Independent recomputation. A transfer-matrix code written from scratch to check this paper (`replication/`; nothing in it is taken from `src/`) reproduces the cylinder values per site to $7.8 \cdot 10^{-20}$ against the extended-precision inputs and to $6.2 \cdot 10^{-17}$ against the float64 ones. It also goes beyond them: both sectors of $w = 26$ and the odd width $w = 27$ in extended precision, and both sectors of $w = 28$ in float64. The new differences d_{26} and d_{27} agree with the L1 predictions made at $W = 24$ (midpoints) and $W = 25$ (odd widths) to $1.9 \cdot 10^{-4}$ and $1.5 \cdot 10^{-4}$ relative, and d_{28}, whose float64 inputs resolve it only to about $6 \cdot 10^{-3}$, to $5.4 \cdot 10^{-4}$; L1 on the odd widths up to $W = 27$ gives 0.5655119110413253187, $1.1 \cdot 10^{-18}$ from the centre above. These widths test the extrapolation outside its sample; they are not used in the estimate.

4. Rigorous bounds

Theorem 4.1 (computer-assisted)**.** $0.545343749727 \le \underline{s} \le \bar{s} \le 0.582767477484$.

For comparison, the Kahn–Lovász bound (the number of perfect matchings of a graph is at most $\prod_v (d_v!)^{1/(2d_v)}$; its authors never published their proof, and a proof is given in [1]), with all degrees $d_v \le 8$, gives $\bar{s} \le \log(8!)/16 = 0.662787681422$.

Trust base. The proof of Theorem 4.1 is computer-assisted, and its correctness rests on two things beyond the mathematics below. First, the certificates are evaluated in IEEE-754 binary64 arithmetic with round-to-nearest (no fused multiply-add contraction, no flush to zero), and their validity follows from the rounding analysis of §4.3. Second, the transfer matrix must be the right matrix: the code is argued correct and validated against brute-force enumeration on small boards (§4.3, §7), but it has not been formally verified.

Both bounds of Theorem 4.1 come from strips of width $w = 24$; the upper bound uses the run-weighted transfer of Lemma 4.6. Table 3 shows how the bounds improve with w. Since $\log \lambda_w/w = s + g(w)/w$ with $g(w) \to -2\sigma$ (§5), the lower bound approaches s like $2\sigma/w = 0.484/w$; the upper bound with the uniform weight $\sqrt{\varphi}$ approaches it, empirically, like $0.497/w$, and the run-weighted one like $0.414/w$.

TABLE 3. Certified bounds from strips of width w: the lower bound, the upper bound with the uniform side weight $\sqrt{\varphi}$ (Lemma 4.7) and the run-weighted upper bound (Lemma 4.6).

w	lower bound on $\underline{s}$	upper, weight $\sqrt{\varphi}$	upper, run weights
4	0.445893077494	0.689765394232	0.669090528409
8	0.505014525148	0.627626147026	0.617278850986
12	0.525175645897	0.606921301094	0.600023045982
16	0.535259669620	0.596568952598	0.591395260725
20	0.541310117470	0.590357544276	0.586218590772
24	0.545343749727	0.586216605404	0.582767477484

4.1. **Lower bound.**

Lemma 4.2 (superadditivity)**.** *For even K and $L = kw + r$ with $0 \le r < w$: $Z(K, L) \ge Z(K, w)^k$.*

Proof. Perfect matchings of k strips $B(K, w)$ placed side by side, completed by vertical dominoes in the last r columns (K is even), are distinct perfect matchings of $B(K, L)$. □

Lemma 4.3 (Collatz–Wielandt, lower)**.** *Let $A = T^2$ be the free-strip transfer of width w, so that $Z(2j, w) = (A^j e_0)_0$. Let $v \ge 0$ with $v_0 > 0$, $\operatorname{supp} v \subseteq \operatorname{supp} A^n e_0$ for some n, and $Av \ge c\,v$ componentwise. Then $Z(2(n+j), w) \ge c^j v_0/C'$ for all j, where $C' = \max_{i \in \operatorname{supp} v} v_i/(A^n e_0)_i < \infty$. In particular $\lambda_w^2 \ge c$.*

Proof. $v \le C' A^n e_0$ componentwise and $A \ge 0$, so $Z(2(n+j), w) = (A^j A^n e_0)_0 \ge (A^j v)_0/C' \ge c^j v_0/C'$. □

Proof of the lower bound. By Lemma 4.2, for even K,

$$\frac{\log Z(K, L)}{KL} \ge \frac{\lfloor L/w \rfloor}{L} \frac{\log Z(K, w)}{K} \longrightarrow \frac{\log \lambda_w}{w} \qquad (K, L \to \infty);$$

for odd K (then L is even) exchange the roles of the sides. So $\underline{s} \ge a_w/w$ for every w, and $\underline{s} \ge s^*$. The certificate of §4.3 gives $a_{24}/24 \ge \frac{1}{48} \log c_{24} \ge 0.545343749727$. □

4.2. **Upper bound.** Cut $B(K, kw)$ into k strips $S^{(1)}, \dots, S^{(k)}$ of width $w \ge 2$ (columns $jw, \dots, jw + w - 1$). King edges change the column by at most one, so every edge of a perfect matching X lies inside a strip or joins column $jw - 1$ to column jw (an edge across cut j). Let A_j be the rows of the cells of column $jw - 1$ matched across cut j (right side of $S^{(j)}$), B_{j+1}

the rows of the cells of column jw matched across cut j (left side of $S^{(j+1)}$), and $B_1 = A_k = \emptyset$. Let $N(B, A)$ be the number of perfect matchings of

$$B(K, w) \setminus \big(\{(b, 0) : b \in B\} \cup \{(a, w-1) : a \in A\}\big),$$

and $M(A, B)$ the number of perfect matchings of the bipartite graph on $A \sqcup B$ with $a \sim b$ iff $|a - b| \le 1$.

Lemma 4.4 (cut identity)**.** *$Z(K, kw) = \sum \prod_{j=1}^{k} N(B_j, A_j) \prod_{j=1}^{k-1} M(A_j, B_{j+1})$, the sum over all $A_1, B_2, A_2, \dots, B_k$ with $B_1 = A_k = \emptyset$.*

Proof. Since $w \ge 2$ the sets are well defined. The restriction of X to strip j is a perfect matching of $S^{(j)} \setminus (B_j \cup A_j)$, and its restriction to cut j a perfect matching between A_j and B_{j+1}; conversely every such tuple of pieces is a perfect matching of $B(K, kw)$. □

The same proof applies to strips of unequal widths, and to a last strip of width 1, whose cells can only be matched across its left cut (there is no cut to its right); the proof of the upper bound uses this for the remainder strip.

Lemma 4.5 (crossing matchings)**.** *If $|A| = |B| = n$ then $M(A, B) \le F_{n+1} \le \varphi^n$, where $F_1 = F_2 = 1$ are the Fibonacci numbers and $\varphi = (1 + \sqrt{5})/2$; $M(A, B) = 0$ if $|A| \ne |B|$.*

Proof. Let $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$ and $M(n) = \max M(A, B)$. *a_1 is matched to b_1 or b_2*: if $a_1 \sim b_j$ with $j \ge 3$, then $b_1 \le b_3 - 2 \le a_1 - 1$, while every $a \ne a_1$ has $a \ge a_1 + 1 \ge b_1 + 2$, so b_1 has no partner. *If $a_1 \sim b_2$*, then $b_1 \le b_2 - 1 \le a_1$ needs a partner $a_i \ge a_1 + 1$ with $a_i \le b_1 + 1 \le a_1 + 1$, so $a_i = a_1 + 1$, $b_1 = a_1$, and the pair $a_2 b_1$ is forced. Thus every matching contains $a_1 b_1$ (the rest is a matching of $n - 1$ against $n - 1$ cells) or both $a_1 b_2$ and $a_2 b_1$ (the rest is $n - 2$ against $n - 2$), so $M(n) \le M(n-1) + M(n-2)$ with $M(0) = M(1) = 1$, i.e. $M(n) \le F_{n+1}$. Finally $F_{n+1} \le \varphi^n$ by induction. □

Equality holds when $A = B$ is an interval. The same branching gives more when A is not an interval. A *run* of a set S of rows is a maximal block of consecutive rows in S; put $F(S) = \prod_{\text{runs } R \text{ of } S} F_{|R|+1}$.

Lemma 4.6 (crossing matchings, run form)**.** *$M(A, B) \le \min(F(A), F(B))$, with equality $M(S, S) = F(S)$. Moreover $F_2 = 1$ and $F_{l+1} \le 2\varphi^{l-2}$ for $l \ge 2$.*

Proof. The relation $a \sim b$ is symmetric, so it suffices to show $M(A, B) \le F(A)$, by induction on $n = |A| = |B|$ (the case $n = 0$ is trivial, and $M = 0$ if $|A| \ne |B|$). Let a_1 be the smallest row of A and l the length of its run. By the proof of Lemma 4.5, every matching contains $a_1 b_1$, or contains $a_1 b_2$ and $a_2 b_1$ with $a_2 = a_1 + 1 \in A$; the second case needs $l \ge 2$. Removing a_1 shortens the first run to $l - 1$, removing a_1, a_2 shortens it to $l - 2$ (a run of length 0 disappears, and $F_1 = 1$), and the other runs are unchanged; write R for the product of their factors. By induction $M(A, B) \le F_l R + F_{l-1} R = F(A)$ if $l \ge 2$, and $M(A, B) \le F(A \setminus \{a_1\}) = F(A)$ if $l = 1$. For $A = B = S$, two runs of S are at distance at least 2, so no edge joins different runs, and on a run of length l the count is F_{l+1} (Lemma 4.5). Finally $F_3 = 2$, $F_4 = 3 \le 2\varphi$ and $F_{l+2} = F_{l+1} + F_l \le 2\varphi^{l-3}(\varphi + 1) = 2\varphi^{l-1}$. □

Hence $M(A, B) \le \sqrt{F(A)F(B)} \le h(A)\, h(B)$ for every weight h with $h(S)^2 \ge F(S)$. We use weights that are products over the runs of S of per-cell factors depending on the position in the run: $1, \sqrt{2}, \sqrt{\varphi}, \sqrt{\varphi}, \dots$, so that a run of length l gets $h^2 = 1, 2, 2\varphi, 2\varphi^2, \dots$, which is at least F_{l+1} by Lemma 4.6. The uniform weight $h(S) = \varphi^{|S|/2}$ of Lemma 4.5 is the special case of one factor $\sqrt{\varphi}$ per cell; it overpays on short runs, and for $A = B = \{0, 2, 4, \dots\}$, a set of runs of length 1, it gives $\varphi^{|S|}$ where $F(S) = 1$. The smallest admissible weight is $h = \sqrt{F}$ itself, with

factors $\sqrt{F_{t+2}/F_{t+1}}$ for the t-th cell of a run; it needs the full run length as a state, and the three factors above truncate it after the second cell.

Put $z = \sqrt{\varphi}$ and

$$W(K,w) = \sum_{A,B} N(B,A)\, z^{|A|+|B|},$$

the weighted number of matchings of a strip in which a cell of the first or last column may be "covered from outside" at weight z.

Lemma 4.7 (strip inequality)**.** *$Z(K,kw) \le W(K,w)^k$. More generally $Z(K,kw) \le W_h(K,w)^k$ with $W_h(K,w) = \sum_{A,B} N(B,A)\, h(A)\, h(B)$, for every h with $h(S)^2 \ge F(S)$.*

Proof. In Lemma 4.4 replace $M(A_j, B_{j+1})$ by $\varphi^n = z^{|A_j|} z^{|B_{j+1}|}$ (Lemma 4.5). The factor $z^{|A_j|}$ belongs to strip j and $z^{|B_{j+1}|}$ to strip $j+1$, so each summand is at most $\prod_j N(B_j, A_j) z^{|B_j|+|A_j|}$. Dropping the constraints $|A_j| = |B_{j+1}|$ and $B_1 = A_k = \emptyset$ adds only nonnegative terms, and the sum factorises over the strips. For general h replace φ^n by $h(A_j)\, h(B_{j+1})$ (Lemma 4.6); $h(\emptyset) = 1$. □

The transfer of §2 computes W when a cell in column 0 or $w-1$ may also be covered from outside at weight z: $W(K,w) = e_0^\top T_z^K e_0$, $T_z \ge 0$. For the run weights the transfer carries, for each of the two side columns, the position of the previous row's side cell in a run of outside covers (0: not covered from outside, 1: first cell of a run, 2: a later cell). A side cell covered from outside gets the factor of its position, 1, $\sqrt{2}$ or $\sqrt{\varphi}$, and advances the state; any other move of a side cell resets it to 0. The state space grows by a factor 9, and $W_h(K,w) = \sum_{l,r} (T_h^K e_0)_{(l,r,0)}$ with $T_h \ge 0$. Lemma 4.8 holds for T_h with the same proof, applied to each of the nine final entries: $W_h(2j,w) \le c' C^j$ with $c' = \sum_{l,r} v_{(l,r,0)}/v_0$, a constant independent of j.

Lemma 4.8 (Collatz–Wielandt, upper)**.** *If $A = T_z^2$, $v \ge 0$, $v_0 > 0$ and $Av \le Cv$ componentwise (including $(Av)_i = 0$ where $v_i = 0$), then $W(2j,w) \le C^j$ for all j.*

Proof. $e_0 \le v/v_0$ and $A \ge 0$, so $A^j e_0 \le A^j v/v_0 \le C^j v/v_0$; take the entry 0. □

Proof of the upper bound. Let $K, L \to \infty$ with KL even. If K is odd then L is even and $Z(K,L) \le Z(K+1,L)$ (add a row of horizontal dominoes), so $(KL)^{-1} \log Z(K,L) \le \frac{K+1}{K}\,((K+1)L)^{-1} \log Z(K+1,L)$ and the factor $(K+1)/K$ tends to 1; we may therefore take $K = 2j$. Write $L = kw + r$, $0 \le r < w$. Cutting off a last strip of width r as in Lemma 4.7 gives $Z(K,L) \le W_h(K,w)^k W_r(K)$ (h the uniform or the run weight), where W_r counts the configurations of a width-r strip with weighted outside covers; crudely $W_r(K) \le (5 + 2\sqrt{2})^{Kr}$ (per cell: already covered, one of at most four later partners, or one of at most two outside covers, each of weight at most $\sqrt{2}$). By Lemma 4.8, with the constant $c' \ge 1$ above for the run weights ($c' = 1$ for $\sqrt{\varphi}$),

$$\frac{\log Z(K,L)}{KL} \le \frac{k}{L}\,\frac{\log C_w}{2} + \frac{k \log c'}{KL} + \frac{w \log(5+2\sqrt{2})}{L} \longrightarrow \frac{\log C_w}{2w}.$$

The certificate of §4.3 at $w = 24$ gives $\bar{s} \le 0.586216605404$ with the weight $\sqrt{\varphi}$ and $\bar{s} \le 0.582767477484$ with the run weights. □

4.3. **Floating-point certificates.** The certificates use float64 vectors v produced by power iteration; how v was obtained does not matter. For the lower bound v is the normalised iterate $A^n e_0$; floating point never creates a positive entry from an exact zero (all terms are nonnegative), so $\operatorname{supp} v \subseteq \operatorname{supp} A^n e_0$. The code computes $y = \mathrm{fl}(Av)$, asserts $v_0 > 0$, $v_i > 10^{-280}$ on $\operatorname{supp} v$ and (upper bound) $y_i = 0$ off $\operatorname{supp} v$, and sets

$$c = \min_{i \in \operatorname{supp} v} \mathrm{fl}(y_i/v_i)\,(1 - 10^{-12}), \qquad C = \max_{i \in \operatorname{supp} v} \mathrm{fl}(y_i/v_i)\,(1 + 10^{-12}).$$

Each cell step writes every entry as a sum of at most six nonnegative terms (already covered, outside cover times z, right, down-left, down, down-right), so over the $2w$ steps of A, $\mathrm{fl}(Av)_i \in (Av)_i\,[(1-u)^{12w}, (1+u)^{12w}]$ with $u = 2^{-53}$: a relative error of at most $12wu\,(1+o(1)) = 3.2\cdot 10^{-14}$ at $w = 24$, plus one rounding in the division. All weights are at least 1 (z is rounded up, so $T_{\mathrm{fl}(z)} \geq T_{\sqrt{\varphi}}$ entrywise) and the entries on the support exceed 10^{-280}, so there is no underflow and a float entry vanishes iff the exact one does. Since $10^{-12} \gg 3.2 \cdot 10^{-14} + u$, the inequalities $Av \geq cv$ and $Av \leq Cv$ hold exactly. For the run-weighted transfer with $n = 3$ states per side, an interior cell step is unchanged; at the left side column an entry of run state 0 collects at most $4n$ terms, and at the right one at most $3n$, while an entry of run state 1 or 2 is reached only by outside covers and needs at most two products by a factor and one addition. Over the $2w$ steps of A this is at most $2\big(4(w-2) + (4n-1) + (3n-1)\big) = 8w + 14n - 20 \leq 12w$ roundings for $w \geq 6$, the same budget. The factors are rounded up to floats, so the float transfer dominates the exact one entrywise, and the factor 1 is exact. The logarithms are taken with 30-digit arithmetic and the results are rounded outward to 12 decimals.

Audit. The following were checked in exact arithmetic: the cut identity and both strip inequalities on 58 boards ($K \leq 8$, $kw \leq 12$); $\max M(A,B) = F_{n+1}$ for all A, B in columns of length $m \leq 11$, and the branching step for $m \leq 8$; the transfer with weight z against brute force as a polynomial in z (162 evaluations); and, for $w = 2, \ldots, 12$, the float ratios against exact rational ratios for the same float vector ($|\text{float/exact} - 1| \leq 1.7 \cdot 10^{-16}$). The audit is sensitive: with $z = 0$ the strip inequality fails on all 40 boards with $k \geq 2$. On these small boards $z = 0.5$ would suffice. The proof needs only $h(S)^2 \geq F(S)$ for every S, which $\sqrt{\varphi}$ meets uniformly and the run weights meet with less to spare. For the run weights, also exactly: $M(A,B) \leq \min(F(A), F(B))$ and $M(S,S) = F(S)$ for all A, B in columns of length $m \leq 8$; the strip inequality with the run weights in interval arithmetic on the same 58 boards; the run-weighted transfer against brute force at rational factors (57 evaluations); and the float ratios against exact ones for $w = 2, \ldots, 8$ ($|\text{float/exact} - 1| \leq 1.7 \cdot 10^{-16}$).

5. Surface free energy

For the free strip of width k, $\log Z(K,k) = Ka_k + O(1)$, and

$$g(k) = a_k - sk \longrightarrow -2\sigma,$$

where σ is the free energy per unit length of a free edge. Even k approach the limit from above, odd k from below, and every even value lies above every odd one (Table 4; a_k from extended-precision power iteration for $k = 14, \ldots, 24$ and float64 with a Collatz–Wielandt bracket for $k = 25, 26$, valid by Lemmas 4.3 and 4.8, the latter at $z = 0$).

Table 4. $g(k) = \log \lambda_k - sk$ for free strips.

k	$g(k)$	k	$g(k)$
19	−0.4840358718678495	20	−0.4840358714180326
21	−0.4840358715709653	22	−0.4840358715195378
23	−0.4840358715369125	24	−0.4840358715310018
25	−0.4840358715330068	26	−0.4840358715323304

The odd sequence converges cleanly (all models of §3 agree to $2 \cdot 10^{-16}$): its differences $g(k+2) - g(k)$ shrink by a factor 8.719 per two columns ($k = 19, \ldots, 25$), that is $e^{2\cdot 1.0827}$, the in-sector gap of §6. The even one carries two corrections, this mode and the parity-splitting mode of the free strip, which alternates with period 4 and shrinks by a factor 13.84 per two columns

(§2); and only the two-exponential and Wynn models are stable under the backtest. The stable models give $-2\sigma \in [-0.4840358715325015, -0.4840358715325006]$; the error of s, multiplied by $k = 26$, dominates:

$$-2\sigma = -0.484035871532501 \pm 1.8 \cdot 10^{-14}, \qquad \sigma = 0.242017935766251 \pm 9 \cdot 10^{-15}.$$

The errors of σ and s are correlated: an error δs in s moves the extrapolated -2σ by $-26\,\delta s$, so $\delta\sigma \approx 13\,\delta s$, and quantities that depend on both add their errors linearly.

5.1. **Corner free energy and exact counts.** The exact counts $Z(K, L)$ of the free boards for $K, L \le 24$ (computed modulo ten primes and reconstructed by the Chinese remainder theorem, `src/zexact.py`) test s and σ without eigenvalues and give the next term of the expansion,

$$\log Z(K, L) = sKL - 2\sigma(K + L) + \kappa + o(1).$$

On the squares, $\rho(L) = \log Z(L, L) - sL^2 + 4\sigma L$ converges to κ with differences shrinking by a ratio that drifts from 5.85 to 5.58 over $L = 14, \dots, 24$. A geometric (Aitken) extrapolation at $L = 24$ gives 0.363196256048; it changes by $1.4 \cdot 10^{-10}$ from $L = 22$, and by $2.6 \cdot 10^{-11}$ if the ratio is extrapolated linearly in $1/L$. The errors of s and σ enter as $L^2\,\delta s + 4L\,\delta\sigma = 1.2 \cdot 10^{-12}$, added linearly because they are correlated. Taking twice the larger extrapolation difference,

$$\kappa = 0.36319625605 \pm 3 \cdot 10^{-10}.$$

Rectangles agree: $|\rho(K, L) - \kappa| \le 6.3 \cdot 10^{-8}$ for $\min(K, L) \ge 20$, with the sign alternating with the parity of $\min(K, L)$.

Since A239273 counts the squares, $a(n) = Z(2n, 2n)$,

$$a(n) \sim C\, d^{\,n^2} f^{-n},$$
$$C = e^{\kappa} = 1.437918032, \quad d = e^{4s} = 9.60273202731337, \quad f = e^{8\sigma} = 6.931966132248,$$

with relative error $-2.0 \cdot 10^{-9}$ at $n = 12$ and about a factor 5.6 smaller per step in n. The b-file of A239273 up to $n = 12$, contributed by the author, agrees with these counts term by term. The form, a constant times one exponential in the side and one in the area, is the one McKay found numerically for the hard squares, the independent sets of the $n \times n$ grid [31]. The ratio of the extrapolation differences is still drifting at $L = 24$, so the slowest edge and corner correction is not yet settled; the error bar of κ allows for this by doubling.

The counts alone also give s. The second differences $[\log Z(L + 2, L + 2) - 2 \log Z(L, L) + \log Z(L - 2, L - 2)]/8$ approach s from above, $5.3 \cdot 10^{-9}$ above it at $L = 22$, and their geometric extrapolation is $1.0 \cdot 10^{-10}$ above the value of §3. The mixed differences $[\log Z(L + 2, L + 2) - \log Z(L + 2, L) - \log Z(L, L + 2) + \log Z(L, L)]/4$, in which the edge and corner terms cancel exactly, approach s from below ($1.1 \cdot 10^{-11}$ at $L = 22$), and their extrapolation agrees with it to $5.9 \cdot 10^{-13}$. The square counts of the JavaScript solvers of the repository, an independent code base, agree with these counts where both exist ($L \le 20$).

6. Parity splitting and its decay length

For even w let $\Delta_w = \log \lambda_w^{(0)} - \log \lambda_w^{(1)}$ and $\xi_w = 2/\log(\Delta_{w-2}/\Delta_w)$ (Table 5). A pure exponential $\log \Delta_w = a - w/\xi$ fits $w = 10, \dots, 24$ poorly ($\xi = 1.1931$, rms $1.5 \cdot 10^{-2}$); with a power prefactor, $\log \Delta_w = a - w/\xi + b \log w$ gives $\xi = 1.2323$, $b = -0.433$, rms $9.3 \cdot 10^{-4}$, and extrapolating $1/\xi_w$ linearly in $1/w$ gives $\xi_\infty = 1.2351$ (prefactor exponent $b = -0.467$). With b fixed at $-1/2$, $\xi = 1.2387$ (rms $2.6 \cdot 10^{-3}$). The three estimates span $\xi \in [1.2323, 1.2387]$; we quote $\xi = 1.235 \pm 0.004$.

The length ξ describes the splitting of the two topological sectors, a property of the cylinder; it is not the bulk correlation length. Inside a sector the transfer has a gap: at $w = 20$

TABLE 5. Splitting of the parity sectors on even cylinders.

w	Δ_w	ξ_w	w	Δ_w	ξ_w
10	$1.521094 \cdot 10^{-4}$	1.15438	18	$1.788983 \cdot 10^{-7}$	1.19451
12	$2.782917 \cdot 10^{-5}$	1.17750	20	$3.373229 \cdot 10^{-8}$	1.19878
14	$5.124578 \cdot 10^{-6}$	1.18201	22	$6.389264 \cdot 10^{-9}$	1.20204
16	$9.544544 \cdot 10^{-7}$	1.19001	24	$1.214909 \cdot 10^{-9}$	1.20486

ARPACK gives $\log(\lambda_1/|\lambda_3|) = 1.082727$ per row, with a negative subleading eigenvalue λ_3 (`out/out_crosscheck.txt`), and the convergence of the power iteration gives the same gap at every $w = 20, \ldots, 25$. The bulk correlation length along the cylinder is therefore $1/1.0827 = 0.92$, and the odd-width corrections, which behave like $e^{-w/\xi_{\mathrm{odd}}} w^b$ with $1/\xi_{\mathrm{odd}} \approx 1.07$, are governed by the same gap. The power prefactor w^b of Δ_w is our fitting ansatz: Ioselevich, Ivanov and Feigelman [19] estimate the splitting of the topological sectors only to exponential accuracy, and the power prefactors in [19, 47] belong to other quantities (correlation functions, the columnar order at odd circumference). The exponent $-1/2$ of those prefactors serves only as one trial value of b in the fits above, and we take no value of ξ from there, since theirs belong to other models. All finite-size corrections we see are exponentially small, as expected for a non-critical (gapped) dimer liquid. On the square lattice, which is critical, the corrections are algebraic, $(f_w - G/\pi)w^2 \to \pi/6$, and a polynomial fit in $1/w^2$ to the transfer-matrix values for $w \le 18$ gives G/π only to $6 \cdot 10^{-8}$; the exact free-fermion values give the same digits, so the truncated asymptotic series, not the transfer matrix, limits this check (`out/out_validate.txt`).

7. Checks on solvable lattices

The transfer code has a switch for the square lattice (no diagonals) and the triangular lattice (one diagonal). Against brute-force enumeration of perfect matchings it agrees on all 90 boards tested (king, triangular and square; free strips and cylinders; $w = 3, \ldots, 6$, $h = 1, \ldots, 4$). It reproduces the exact domicule counts of the repository for all square boards up to 18×18, for rectangles, and the 20×20 count.

Square lattice. The free-strip growth rates agree with the product formula $\lambda_w^{\mathrm{sq}} = \prod_{j=1}^{\lfloor w/2 \rfloor} (a_j + \sqrt{1 + a_j^2})$, $a_j = \cos(\pi j/(w+1))$ [20], and both cylinder sectors with the free-fermion formula, to 10^{-16}.

Triangular lattice. The triangular lattice is planar, and we found a Pfaffian orientation by a sign pattern periodic modulo 2, checked against the exact counts on small boards and by $|\mathrm{Pf}| = Z$ on the 8×8, 10×10 and 12×12 boards (1848358752, 1852300296658080, 57261989986381118073728). Averaging the logarithm of the Bloch determinant over a 32×32 and a 64×64 grid of momenta gives the same value,

$$s_{\mathrm{tri}} = 0.428594537464959,$$

and the transfer matrix on the cylinder of circumference 17 gives 0.428594537464956, a difference of $2.9 \cdot 10^{-15}$. This agrees with the literature value $0.857189074\ldots$ per dimer [45]; the orientation goes back to Montroll [27], and the check only tests our code.

Context. Table 6 places s among the three lattices. The Bethe value $\frac{1}{2}\log((d-1)^{d-1}/d^{d-2})$ per vertex is the growth rate of the expected number of perfect matchings of a random d-regular graph [3] and the zeroth order of Nagle's expansion [28]. For d-regular bipartite graphs it is a lower bound [36], and the square lattice lies above it; on the non-bipartite triangular and king lattices it overestimates s, by 2.7% and 1.2%.

TABLE 6. Entropy per site of the three lattices, the Bethe value and the Kahn–Lovász bound $\log(d!)/(2d)$.

lattice	d	s	Bethe	s/Bethe	Kahn–Lovász
square	4	$G/\pi = 0.291560904031$	0.261624072	1.1144	0.397256729
triangular	6	0.428594537465	0.440075843	0.97391	0.548270934
king	8	0.565511911041	0.572360897	0.98803	0.662787681

8. Convexity conjectures and a conditional two-sided bound

The bounds of Theorem 4.1 converge like $1/w$ because every strip has two free sides. Differences of strip growth rates cancel the surface term. For hard squares on the square lattice the strip transfer matrix is symmetric, and the maximum principle turns them into rigorous lower bounds [4] (see also [22]); Pavlov [32] proved that the differences of consecutive widths converge exponentially. In reflection terms the lower-bound argument uses a positive semidefinite gluing form across the reflection line (reflection positivity). Here the gluing form between two half-boards is the number $M(A, B)$ of matchings across the cut (Lemma 4.5), and it is indefinite: for columns of length 3 its smallest eigenvalue is $1 - \sqrt{2}$, on a vector realised by actual half-boards. Nor do the cylinders give a proven upper bound. At every odd circumference $w = 3, 5, \ldots, 25$ the cylinder value f_w lies below the estimate of s, by $2.5 \cdot 10^{-3}$ at $w = 3$ and $6.7 \cdot 10^{-15}$ at $w = 25$ (Table 1 lists $w \geq 13$); on even cylinders the sector $p = 0$ lies above it at every width we computed, but we have no argument that makes it a bound. The comparison is with the estimate, so this is evidence, not a proof. We state the two sign properties that the data show instead.

Conjecture 8.1. *Let $a_0 = a_1 = 0$ and $a_k = \log \lambda_k$ for $k \geq 2$.*

(L_∞) *$q \mapsto a_{2q}$ is convex: $a_{2q+2} - 2a_{2q} + a_{2q-2} \geq 0$ for all $q \geq 1$.*

(O) *$a_{k+1} - a_k \geq s$ for every odd k.*

Proposition 8.2. *(L_∞) implies $s \geq 0.565511911035$, and (O) implies $s \leq 0.565511911049$.*

Proof. By §2 and the proof of the lower bound of Theorem 4.1, $s = \lim a_k/k$. Under (L_∞) the increments $a_{2q+2} - a_{2q}$ are nondecreasing; since $a_{2q}/(2q) \to s$, they increase to $2s$, so $s \geq (a_{24} - a_{22})/2$. Under (O), $s \leq a_{24} - a_{23}$. The certificates of §4.3 give, with outward rounding,

$$\log \lambda_{24} \geq 13.08824999346029, \qquad \log \lambda_{23} \geq 12.52273808241305,$$
$$\log \lambda_{22} \leq 11.95722617139012, \qquad \log \lambda_{24} \leq 13.08824999346131;$$

the lower bounds are 24 and 23 times the certified per-site lower bounds at $w = 24, 23$ as stored in `data/bounds.json` (before the outward rounding to 12 decimals of Table 3), minus 10^{-14} for their storage as floats, and the upper bounds come from Lemma 4.8 with $z = 0$. Hence $s \geq (13.08824999346029 - 11.95722617139012)/2 = 0.565511911035\ldots$ and $s \leq 13.08824999346131 - 12.52273808241305 = 0.565511911048\ldots$, rounded outward. □

Under both hypotheses s is known to ten digits, 0.5655119110. The estimate of §3 lies $6.3 \cdot 10^{-12}$ above the lower and $7.7 \cdot 10^{-12}$ below the upper conditional bound. Only the tails are used: the lower bound needs $a_{24} - a_{22} \leq 2s$, which is the Calkin–Wilf inequality that reflection positivity would give, and the upper bound needs (O) at $k = 23$. The unconditional bounds remain those of Theorem 4.1.

Evidence. Both statements hold wherever we can compute: the second differences of a_{2q} are positive for $2q = 2, 4, \ldots, 24$, those of the odd widths are negative for $k = 3, 5, \ldots, 23$, and the increments $a_{k+1} - a_k - s$ alternate in sign for $k = 0, \ldots, 25$; at $k = 23$ the margin of (O)

is $5.9 \cdot 10^{-12}$. One mode explains all of this. The ratio of consecutive increments tends to $-0.338 = -e^{-1.083}$, the in-sector gap of §6, so

$$g(k) = a_k - sk \approx -2\sigma + C(-\rho)^k, \qquad \rho = e^{-1.083},$$

and a single sign, $C > 0$, gives both (L_∞) and (O) for large k. A second, faster mode is the splitting of the two parity sectors of the free strip, which alternates with period 4 and decays by a factor -13.84 per two columns (§2). In the second differences of a_{2q} the two modes are $0.1147^q = \rho^{2q}$ and $(-0.072)^q$, and at $2q = 24$ the second is 0.007 of the first, shrinking by a factor 0.63 per step.

The convexity is a quantitative property of unit weights, not a structural one. Its finite form $Z(2q, m)^2 \le Z(2q-2, m)Z(2q+2, m)$ holds in all 1453 cases we tested on the king's graph, but it fails on the triangular lattice, for every unequal pair of diagonal weights we tried, and for equal diagonal weights $x = \frac{3}{2}, 2, 3$ on wide strips; along equal weights the two rates above cross, on our data, between $x = \frac{7}{4}$ and $x = 2$. The companion note [33] collects these tests and the counterexamples, explains the onsets for nearly equal diagonal weights at $m = 4$ and 8 by the Krein signature of the left–right reflection, and lists the obstructions to the standard proof techniques.

Open problem 8.3. Prove (L_∞) and (O) at unit weights, i.e. the sign $C > 0$ with a bound on the remainder for every $k \ge 22$; Proposition 8.2 would then give ten rigorous digits. Another route to ten digits is the sector bracket $f_w^{(1)} \le s \le f_w^{(0)}$ on even cylinders, which holds in all our data (§2) and has width $5.06 \cdot 10^{-11}$ at $w = 24$.

9. Reproducibility

All code and outputs are in the repository

`https://github.com/dawidrylko/lattice-dimers`,

tag `king-entropy-v1.0.0`, directory `research/king-entropy/`. The scripts need Python 3.13 with numpy 2.5.3 [13], mpmath 1.4.1 [39] and scipy 1.18.1 [40] (whose sparse eigensolver is ARPACK [21]), as pinned in `src/requirements.txt`. Every number in this paper is listed in `paper/NUMBERS.md`, with the committed output file it is read from and the command that wrote that file; the captions and the text name the main ones. Many outputs record the commit of the code that wrote them. The first version of this work is the tag `king-entropy-v0.1.0`, where the files sit directly in `research/king-entropy/`; the commits recorded in the outputs of that version refer to its history, which the tag keeps. The main commands, each run as `python3` followed by the script and its arguments from `research/king-entropy/`:

script	output
`src/tm2.py king cyl 3 13, ... 14 19`	`data/sec/sec_king_cyl_*.json`
`src/cylw.py king cyl` w p, $w = 20, \dots, 25$	`data/eig/eig_king_cyl_w*_p*.json`
`src/estimate.py`	`out/out_estimate.txt`, `data/z2.json`
`src/lean.py king cyl` w p `longdouble`	`data/lean/lean_*.json`
`src/errbar.py`	`out/out_errbar.txt`, `data/errbar.json`
`src/bounds.py 24`	`out/out_bounds.txt`, `data/bounds.json`
`src/runweight.py 3state` w`...`	`out/out_runweight_3state.txt`
`src/audit_bounds.py`	`out/out_audit_bounds.txt`
`src/pfaff_rigor.py`	`out/out_pfaff.txt`
`src/corner.py`	`out/out_corner.txt`, `data/corner.json`
`src/surface.py`	`out/out_surface.txt`
`src/cw.py`	`out/out_cw.txt`
`src/zexact.py ...`; `src/ltest.py`	`data/zex/zex_*.json`, `out/out_ltest.txt`
`src/lcert.py 25 30, ... 26 30`	`out/out_lcert_m25.txt`, ...
`src/lweight.py [big]`	`out/out_lweight*.txt`
`log-convexity/symmetry.py`	`log-convexity/out_symmetry.txt`
`log-convexity/linf.py 16`	`log-convexity/out_linf.txt`
`log-convexity/involution.py`	`log-convexity/out_involution.txt`
`log-convexity/modes.py`	`log-convexity/out_modes.txt`
`src/validate.py`; `src/crosscheck.py`	`out/out_validate.txt`, ...
`src/derived.py`	`out/out_derived.txt`

The outputs were produced on two x86-64 machines with four cores and about 16 GiB of memory each: the first version on one, the runs added in this version (extended precision, run weights, the corner counts, the Pfaffian certificate and the two-mode fits) on the other. The extended-precision runs use the 64-bit mantissa of the x87 `long double`, which `src/lean.py` asserts. The algorithms are those of §2; the implementation applies each cell step to reshaped views of the state vector, without forming the transfer matrix. The widest runs dominate the cost: the cylinder $w = 25$ (8.4 GiB, 22.5 min), the certified bounds at $w = 24$ (17 min), the run-weighted bound at $w = 24$ (13.3 GiB including the memory-mapped vector, 63 min), the exact counts of the 24×24 board (1.5 GiB, 52 min), and the certified (L) tests at $m = 26$ (8.0 GiB, 43 min). The script `src/repro.py` re-runs `src/estimate.py`, `src/cw.py`, `src/bounds.py 16` and `src/cylw.py king cyl 20 0` from a clean export of the repository and checks that their outputs are identical up to timings and commit tags.

Data availability statement

The code and data that support the findings of this paper are openly available in the repository `https://github.com/dawidrylko/lattice-dimers` at the tag `king-entropy-v1.0.0`, and are archived on Zenodo, DOI: 10.5281/zenodo.23686293.

Disclosure statement

The author reports no competing interests.

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