LOG-CONVEXITY OF STRIP DIMER COUNTS ON THE KING'S GRAPH

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ABSTRACT. Let $Z(k,m)$ be the number of perfect matchings of the $k \times m$ king board, the square grid with both diagonals, i.e. the number of its domicule tilings. In all 1453 cases we tested, on free strips of width $m \le 26$ and cylinders of circumference $m \le 22$, the sequence $q \mapsto Z(2q,m)$ is log-convex; the tests are exact or carry a rigorous rounding bound. The property is not structural. It fails on free strips of the triangular lattice, for every unequal pair of diagonal weights we tried (for nearly equal weights at $m = 4$ and 8 a Krein-signature argument for the left–right reflection predicts the onset exactly), and on wide strips for equal diagonal weights $x = \frac{3}{2}, 2, 3$. Its limit, convexity of $q \mapsto \log \lambda_{2q}$ for the strip growth rates, would give a conditional lower bound on the entropy with ten correct digits. Along equal weights it is a competition between two modes of the transfer matrix whose rates cross, on our data, between $x = \frac{7}{4}$ and $x = 2$. We collect the obstructions to the standard proof techniques and two exact reformulations.

1. INTRODUCTION

A *domicule* is a pair of cells of the square grid that share a side or a corner, and a domicule tiling of a board is a perfect matching of the *king's graph* on its cells: cells are adjacent when they are at chessboard king distance 1. Let $Z(k,m)$ be the number of tilings of the board with k rows and m columns (OEIS A239264 [2]), and $Z_{\mathrm{cyl}}(k,m)$ the same number when column $m-1$ is also adjacent to column 0. The companion paper [4] computes the entropy per site $s = \lim \log Z(K,L)/(KL)$,

$$s = 0.5655119110413253 \pm 6 \cdot 10^{-16} \quad \text{(not rigorous)},$$
$$0.545343749727 \le s \le 0.582767477484 \quad \text{(rigorous)}.$$

The rigorous bounds come from strips with free sides and converge only like $1/w$ in the width w. For hard squares on the square lattice, the symmetric strip transfer matrix gives the Calkin–Wilf lower bounds [1], differences of logarithmic strip growth rates divided by p, and Pavlov [3] proved that the differences of consecutive widths converge exponentially, hence so do these bounds. For the king's graph the same bounds would read $(a_{2q+p} - a_{2q})/p \le s$, where $a_k = \log \lambda_k$ and λ_k is the growth rate of the free strip of width k, $\lambda_k^2 = \lim_j Z(2j,k)^{1/j}$. In reflection terms the argument uses a positive semidefinite gluing form across the reflection line (reflection positivity), which fails for the king's graph across lines between rows or columns (§5). The weakest hypothesis that still gives the even-p bounds is

(L_∞) $\qquad$ (L_∞) $\qquad q \mapsto a_{2q}$ is convex,

since a convex sequence with $a_{2q}/(2q) \to s$ has increments that increase to $2s$. Under (L_∞), certified values of λ_{22} and λ_{24} give $s \ge 0.565511911035$ [4, Prop. 8.2]. This note is about the

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finite statement behind it,

(L) (L) $Z(2q,m)^2 \le Z(2q-2,m)\,Z(2q+2,m)$ for all $m \ge 1,\ q \ge 1$.

(L) implies (L_∞): it makes $q \mapsto m^{-1}\log Z(2q,m)$ convex for every m; since $Z(2q,m) = Z(m,2q)$, these sequences tend to a_{2q} along even m, and a pointwise limit of convex sequences is convex.

We do *not* conjecture (L). On the king's graph it holds in every case we tested at unit weights (§2), but it is not a structural property of the model: it fails on the triangular lattice, under asymmetric diagonal weights, and for larger equal diagonal weights (§3). The conjecture of the companion paper is (L_∞), and §4 explains why it is a quantitative statement about unit weights.

2. Tests at unit weights

The counts $Z(k,m)$ were computed exactly with the row transfer of [4], reduced modulo 60-bit primes and combined by the Chinese remainder theorem; the number of primes is chosen so that their product exceeds the Kahn–Lovász bound on the number of perfect matchings, so the result is the exact integer. For $m = 25, 26$ we used x87 extended precision (64-bit mantissa) with a rigorous rounding bound: each cell step writes a sum of at most five nonnegative terms, so the computed $Z(k,m)$ is within a factor $(1+u)^{5mk}$ of the true value, $u = 2^{-64}$, and the smallest certified margin is $1.237 \cdot 10^{-10}$. Table 1 lists the tests. (L) holds on free strips of width $m \le 26$ and on cylinders of circumference $m \le 22$ in the stated row ranges. The margin $\log Z(2q-2) + \log Z(2q+2) - 2\log Z(2q)$ decreases monotonically in q for every m, by a ratio tending to 0.16–0.20; there is no dip at intermediate q and nothing that shrinks with m.

Table 1. Tests of (L), $Z(2q)^2 \le Z(2q-2)Z(2q+2)$, at unit weights. The row range is that of the exact counts at each width; it shrinks as the width grows, except at $m = 24$.

lattice, boundary	widths m	rows $2q$	cases	violations
king, free	2–24	≤ 120 to ≤ 20	875	0
king, cylinder	3–22	≤ 80 to ≤ 22	550	0
king, free (x87, certified)	25, 26	≤ 28	28	0
square, free / cylinder	≤ 12	≤ 60	319 / 290	0 / 0
triangular, cylinder	≤ 12	≤ 60	290	0
triangular, free	≤ 12	≤ 60	319	**92**

(L) is specific to even row counts: for even m, the sequence $j \mapsto Z(2j+1,m)$ is log-*concave* in 217 of the 220 cases tested. This matches the free strips of the companion paper, where even widths approach the limit of $a_k - sk$ from above and odd widths from below.

3. Counterexamples

3.1. **The triangular lattice.** On free strips of the triangular lattice (one diagonal per square) (L) fails from $m = 5$ on: $Z(6,5) = 6475$, $Z(8,5) = 210560$, $Z(10,5) = 6842465$ and $210560^2 - 6475 \cdot 6842465 = 30552725 > 0$, confirmed by brute force. An argument for (L) must therefore use the king's graph itself, not only non-planarity or non-bipartiteness.

3.2. **Unequal diagonal weights and the Krein signature.** Give the orthogonal edges weight 1 and the two diagonals weights x_{dl} and x_{dr}. The reflection $J\colon c \mapsto m-1-c$ of a strip of width m exchanges the two diagonals, so it is a symmetry exactly when $x_{\mathrm{dl}} = x_{\mathrm{dr}}$. Write the row transfer as $T = HM$, with H the horizontal dimers inside a row and M the kernel of the edges between two rows. Then $H = H^{\mathsf{T}} = JHJ$ and, for *any* diagonal weights, $M^{\mathsf{T}} = JMJ$:

transposition exchanges the roles of the two rows, which exchanges the two kinds of diagonals, and conjugation by J exchanges them back. Hence $G = JM$ is symmetric and $GT = JMHM$ is symmetric, i.e. T is self-adjoint for the indefinite form G, and $Ge_0 = e_0$ for the empty state e_0.

Lemma 3.1. *Let λ_j be a simple real eigenvalue of T with right eigenvector r_j, and suppose $r_j^{\mathsf{T}} G r_j \neq 0$. Then the weight of λ_j in $Z(k,m) = (T^k)_{00} = \sum_i c_i \lambda_i^k$ is*

$$c_j = \frac{r_j(0)^2}{r_j^{\mathsf{T}} G\, r_j},$$

so the sign of c_j is the Krein signature of r_j.

Proof. From $GT = T^{\mathsf{T}} G$, the vector Gr_j is a left eigenvector for λ_j, so the spectral projection is $P_j = r_j (Gr_j)^{\mathsf{T}} / (r_j^{\mathsf{T}} G r_j)$, and $c_j = (P_j)_{00} = r_j(0)\,(Gr_j)(0)/(r_j^{\mathsf{T}} G r_j)$. Finally $(Gr_j)(0) = e_0^{\mathsf{T}} G r_j = (Ge_0)^{\mathsf{T}} r_j = r_j(0)$. □

We checked the algebra in exact arithmetic for $m \le 7$ and five weight sets, and the formula against a direct eigendecomposition for $m \le 9$ (relative error $7 \cdot 10^{-13}$). With equal weights the empty state is J-even, Z sees only the J-even sector, and there $G = M$; on the J-odd sector $G = -M$. Under the perturbation $(1, 1-\varepsilon)$ the top J-odd level, which is M-positive, becomes visible with a *negative* weight $-K\varepsilon^2$ ($K = 0.70, 1.31, 1.63$ at $m = 4, 5, 8$, constant for $\varepsilon = 10^{-3}, \dots, 3 \cdot 10^{-2}$).

The sign of $Z(2q-2)Z(2q+2) - Z(2q)^2$ for large q follows from the spectrum. Write $Z(2q) = c_1 L_1^q + c_2 L_2^q + \dots$ with $L_i = \lambda_i^2$, where L_1 is the Perron root of the irreducible part of T^2 that e_0 sees, and L_2 the visible level of next largest modulus. If L_2 is real and positive and dominates the rest, then

$$Z(2q-2)Z(2q+2) - Z(2q)^2 \sim c_1 c_2 (L_1 L_2)^{q-1} (L_1 - L_2)^2,$$

and $c_1 > 0$, so (L) fails for large q exactly when $c_2 < 0$. The Perron vector is positive, hence J-even; so a J-odd level can only matter if it lies above every visible level other than the Perron root. Among $m \le 11$ this happens at $m = 4$ and $m = 8$; the onsets were computed for $m \le 10$. The exact counts confirm the prediction: the weights $(1, \frac{9}{10})$ violate (L) at $m = 4$ from $2q = 56$, at $m = 8$ from $2q = 86$ and at $m = 12$ from $2q = 112$, and $(1, \frac{19}{20})$ at $m = 8$ from $2q = 118$. At $m = 4$ and $m = 8$ the onsets agree exactly with the predicted ones; the failure at $m = 12$ lies outside the range of the prediction.

Clearly unequal weights fail much earlier. The weights $(1, \frac{1}{2})$ violate (L) at $m = 5$ (14 of 19 values of q, $2q \le 38$), $m = 8$ and $m = 9$, and at no other $m \le 9$; $(1, \frac{3}{4})$ fails at $m = 9$ (5 of 19 values, first at $2q = 30$). Every pair of unequal diagonal weights on which (L) was tested, namely $(1, \frac{9}{10})$, $(1, \frac{19}{20})$, $(1, \frac{3}{4})$, $(1, \frac{1}{2})$, $(1, \frac{1}{4})$, $(1, \frac{1}{10})$, $(1, 0)$ and $(2, \frac{1}{2})$, violates it at some m. The symmetrised transfer $H(M + M^{\mathsf{T}})/2$ has no violation where we tested it: for $m \le 9$, $2q \le 38$ and the weights $(1, \frac{1}{2})$, $(1, \frac{3}{4})$, $(1, 0)$ and $(1, \frac{9}{10})$, and for $(1, \frac{9}{10})$ at $m = 8$ up to $2q = 198$. This covers every failure of HM in those ranges, but not the counterexamples at $m = 4$ and $m = 12$ or the weights $(1, \frac{19}{20})$.

The orbits of J split the count as $Z = Z_+ + Z_-$ with $Z_\pm = (Z \pm F)/2$, where F counts the J-symmetric tilings. Z_+ satisfies (L) for $m \le 12$; Z_- does not (at $m = 4$ for every q, otherwise only at the trivial $q = 1$, where $Z_-(0) = 0$).

3.3. **Equal diagonal weights.** With equal weights x on both diagonals, J remains a symmetry, and (L) holds for $x \in \{0, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 10, 100\}$ on narrow strips ($m \le 9$, $2q \le 38$), but not on wide ones. Exactly, $Z(8,m)^2 > Z(6,m)\,Z(10,m)$ for every m with $35 \le m \le 60$ at $x = \frac{3}{2}$, with $21 \le m \le 60$ at $x = 2$ and with $19 \le m \le 60$ at $x = 3$, while $x = 1$ and $x = \frac{5}{4}$ give no violation

at $2q = 8$ for $m \le 60$. These finite failures are exact and do not depend on any extrapolation. So the left–right symmetry alone does not imply (L).

They are the finite-width traces of a failure of convexity of the growth rates at small q: for $x \ge \frac{3}{2}$ the second differences $a_{2q+2} - 2a_{2q} + a_{2q-2}$ are negative at $2q = 8$ and $2q = 12$ (at $x = \frac{3}{2}$: $-6.2 \cdot 10^{-4}$ at $2q = 8$), so at these weights (L_∞) fails as stated. Whether the growth rates become convex for large q, which is all that the conditional bound uses, is the subject of the next section.

4. Two modes and the crossing of their rates

At unit weights the second differences of a_{2q} are positive for $2q = 2, \dots, 24$ ($a_0 = 0$), those of the odd widths are negative for $k = 3, \dots, 23$, and the increments $a_{k+1} - a_k - s$ alternate in sign for $k = 0, \dots, 25$ [4, §8]. All of this is the signature of one alternating mode, $a_k - sk \approx -2\sigma + C(-\rho)^k$ with $\rho = e^{-1.083}$ (the in-sector gap of the transfer) and $C > 0$. A second, faster mode comes from the splitting of the two parity sectors of the free strip, which alternates with period 4 in k and decays by a factor -13.84 per two columns. In the second differences $d_q = a_{2q+2} - 2a_{2q} + a_{2q-2}$ the two modes appear as

$$d_q \approx P\rho_+^q + Q\rho_-^q, \qquad \rho_+ = \rho^2 = 0.1147, \quad \rho_- = -0.072,$$

and at $2q = 24$ the second term is 0.007 of the first, shrinking by a factor 0.63 per step. Along equal weights x both rates grow, at different speeds. (L_∞) holds for large q when $\rho_+ > |\rho_-|$ and $P > 0$, and fails when $|\rho_-| > \rho_+$. We define x^* as the weight at which the two rates cross, $\rho_+(x^*) = |\rho_-(x^*)|$.

Table 2. Two-mode Prony fits of the second differences d_q of the free strips with both diagonal weights x, widths $2q \le 22$. The ratio $|\rho_-|/\rho_+$ is given for the first and the last of four sliding windows (window $2q = 8..14$ and $2q = 14..20$); the last column lists the even widths $2q \le 20$ with $d_q < 0$.

x	ρ_+	ρ_-	$\lvert\rho_-\rvert/\rho_+$ first	last	dominant	$d_q < 0$ at $2q =$
1	0.1147	−0.0718	0.670	0.627	positive	none
5/4	0.1433	−0.1217	0.873	0.849	positive	none
3/2	0.1710	−0.1611	0.971	0.942	positive	8, 12, 16, 20
7/4	0.1971	−0.1938	1.019	0.984	positive	8, 12, 16, 20
2	0.2214	−0.2219	1.047	1.002	alternating	8, 12, 16, 20
3	0.3027	−0.3082	1.095	1.018	alternating	8, 12, 16, 20

Table 2 gives the fits (`log-convexity/modes.py`, float64 power iteration to a relative tolerance 10^{-15}; the smallest $|d_q|$ used, $7.6 \cdot 10^{-10}$ at $x = 1$, is five orders of magnitude above the error of the growth rates). At $x = 1$ the fitted $\rho_+ = 0.1147$ and $\rho_- = -0.072$ agree with the two rates read off the unit-weight data above. Three features matter.

At $x = 3/2$ the asymptotic sign is unresolved and leans convex. The second differences are negative at $2q = 8, 12, 16, 20$ and positive in between; the last negative value is $-3.6 \cdot 10^{-9}$ at $2q = 20$. In the two-mode model the positive mode wins, $|\rho_-|/\rho_+ = 0.942$, but slowly: with the amplitudes of the last window the alternating term is 1.34 times the positive one at $2q = 14$ and still 0.995 times at $2q = 24$, the next width at which the alternating term is negative. That margin is smaller than the drift of the fitted ratio over the windows, from 0.971 to 0.942: the margin needs a ratio below 0.943, and the ratio of every earlier window, 0.945 to 0.971, would make $d_q < 0$ at $2q = 24$. So the model decides neither the sign at $2q = 24$ nor from which $2q$ on

$d_q > 0$. This is an extrapolation from four windows, not a computation; what is exact at $x = 3/2$ are only the finite-strip failures of Section 3.3.

The ratio drifts downwards with the window at every x, by 0.03 to 0.08 from the first to the last window. At $x = 2$ and $x = 3$ the last ratios are 1.002 and 1.018, so the alternating mode dominates on the data, and (L_∞) would fail for large q if these ratios are the limits. That failure is plausible, but it is an extrapolation too: a further drift of the same size would bring both ratios below 1.

The crossing. Linear interpolation of $\rho_+ - |\rho_-|$ between $x = 7/4$ and $x = 2$ gives $x^* \approx 1.96$ from the last windows. Because of the drift this is a lower estimate of x^* rather than a value; on the present data $x^* \in (7/4, 2)$, and widths beyond 22 would be needed to locate it better. Nothing in the data suggests a crossing at $x \le 3/2$.

The large-x limit is suggestive. Diagonal steps preserve the colour of the checkerboard, so at $x = \infty$ the model decouples into two square-lattice dimer models, one on each colour class (each class with its diagonal edges is a square lattice turned by 45°), which are critical. The alternating mode that takes over above x^* may be a precursor of this decoupled limit. We do not know whether x^* marks a phase transition or only a crossover in the subleading spectrum. Either way, $x = 1$ lies below x^*, and a proof of (L_∞) at $x = 1$ has to see the size of the diagonal weight; this rules out arguments that are uniform in x.

5. Obstructions to standard proofs

Reflection positivity. Reflect through a line between two rows. The reflection exchanges the two diagonals, and the form that glues a top half to a bottom half is the number $M(A, B)$ of matchings between the sets A, B of cells matched across the line, with $a \sim b$ iff $|a - b| \le 1$. This form is indefinite: for columns of length $m = 3$ its smallest eigenvalue is $1 - \sqrt{2}$, with eigenvector $c = (1, -\sqrt{2}, 1)$ on $A = \{0\}, \{1\}, \{2\}$. Reflection positivity would need $\langle F, \theta F\rangle \ge 0$ for every function F of the top half. Take $F = \sum_A (c_A/g(A))\,[\text{boundary state} = A]$, where $g(A) > 0$ is the number of three-row half-boards with boundary state A (here $10, 4, 10$); then $\langle F, \theta F\rangle = c^{\mathsf{T}} M c < 0$. The vector c is J-even, so restricting to the sector that Z sees does not help: on the J-even sector M has $1, 1, 3, 8, 20, 34, 93, 186$ negative eigenvalues for $m = 3, \dots, 10$. By the symmetry of the lattice the same holds for lines between columns. For a reflection through a line of vertices the gluing kernel is $[U \ne D]$, which is indefinite (eigenvalues ± 1 already for one vertex); we have not checked whether its negative directions are realised. At equal weights $T = HM$ with H and M symmetric, so the obstruction is the indefiniteness of M, not the asymmetry of T.

Moment sequences. If $q \mapsto Z(2q, m)$ were a Stieltjes moment sequence of a positive measure, (L) would follow; it is not. The eigenvalues of T^2 that e_0 sees are non-real or real negative from $m = 3$ on (small: at most $0.047\,\lambda_1^2$ in modulus at $m = 10$), and the Hankel matrices $[Z(2(i + j), m)]$ and $[Z(2(i + j + 1), m)]$ have negative minors for every free width $m \ge 3$ and every circumference $m \ge 7$. (L) is only the consecutive 2×2 part of this.

Total nonnegativity of the transfer, in any ordering of the states, would force a real nonnegative spectrum, and fails for the same reason. The Lindström–Gessel–Viennot lemma needs a planar, acyclic path model that the crossing diagonals do not provide. *Interlacing* of the strip spectra would need a Hermitian structure, or a Pontryagin space of small negative index, but G has exponentially many negative directions. Monotonicity of the gap gives the decay rate of the second differences, not their sign.

6. Reformulations

Parity of paths. Superpose a matching X of rows $[0, 2q)$ with a matching Y of rows $[2, 2q + 2)$ ($Z(2q)^2$ pairs), or X' of $[0, 2q + 2)$ with Y' of $[2, 2q)$ ($Z(2q + 2)Z(2q - 2)$ pairs). The union H has

degree one on the top rows $\{0,1\}$ and the bottom rows $\{2q,2q+1\}$ and degree two in between, so it consists of paths and even cycles. In the first case top–top and bottom–bottom paths have odd length and top–bottom (TB) paths even length; in the second all paths are odd. Each H has $2^{c(H)}$ labelings, $c(H)$ the number of cycles of length at least four, so

$$Z(2q-2)Z(2q+2)-Z(2q)^2=\sum_{H\colon t(H)\geq 1}2^{c(H)}([\text{all TB paths odd}]-[\text{all TB paths even}]),$$

where $t(H)$ is the number of TB paths. We checked this identity pair by pair for small (m,q), up to $3.4\cdot 10^6$ pairs. The odd configurations win separately for every t (e.g. $m=5$, $q=2$: for $t=2$, 1847660 against 912656; for $t=4$, 32616 against 0). So (L) asks for a sign-reversing injection that flips the parity of all top–bottom paths, and by §3 it would have to depend on the lattice and on the size of the diagonal weight. With a shift of one row instead of two, the same identity computes $Z(2q+1)Z(2q-1)-Z(2q)^2$, which is negative for even m; there the even configurations win.

A Krein-space form. Cut the $2q$-row board in the middle and reflect the bottom half (the lattice is invariant under $r\mapsto -r$). Then $Z(k_1+k_2)=u_{k_1}^{\mathsf{T}}Mu_{k_2}$, with u_k the boundary vector of a k-row half-board and M the symmetric, indefinite gluing form above. Since $Z(2q\pm 2)=u_{q\pm 1}^{\mathsf{T}}Mu_{q\pm 1}>0$, (L) says exactly that M is positive semidefinite on the span of u_{q-1} and u_{q+1}: the Gram determinant of the pair is $Z(2q-2)Z(2q+2)-Z(2q)^2$. The Hankel matrices of §5 are the M-Gram matrices of the longer families $\{u_0,u_2,u_4,\dots\}$ and $\{u_1,u_3,\dots\}$, on which M is not positive.

Partial involutions. Swapping the parts of two superposed matchings below the first row where their boundaries agree (plainly or after the reflection J) is an involution between the two types of pairs, preserves the number of diagonal dimers, and accounts for part of the difference. It leaves an unmatched remainder that does not shrink with q: the ratio of the two unmatched counts tends to 1 (from 0.26 to 0.98 at $m=3$ and from 0.54 to 0.99999 at $m=5$, for $q=2,\dots,8$). Together with the flip of the pairs without top–bottom paths, it reduces the even-type pairs with $t\geq 1$ only from 194 to 104 ($m=3$) and from 912656 to 861364 ($m=5$). With asymmetric weights the twisted swap breaks the identity, as it must.

7. Open problems

Open problem 7.1. Prove (L_∞) at unit weights, or its consequence $a_{24}-a_{22}\leq 2s$. By §4 this amounts to the sign $C>0$ of the amplitude of the mode $C(-\rho)^k$ in a_k-sk, with the faster alternating mode and the remainder controlled for every even $k\geq 22$.

Open problem 7.2. Locate x^* more precisely, and decide whether $q\mapsto a_{2q}$ is convex for large q at $x=\frac{3}{2}$, where the two-mode fits favour the bulk mode, and whether its second differences change sign infinitely often at $x=2$ and $x=3$, where they favour the alternating mode. Both need growth rates of wider strips than those used here.

Reproducibility

All scripts and outputs are in the repository `https://github.com/dawidrylko/lattice-dimers`, directory `research/king-entropy/`; every number of this note is mapped to its output file in `paper/NUMBERS.md`. The tests of §2 are `src/zexact.py`, `src/ltest.py`, `src/lcert.py` and `src/cw.py`; the weights of §3 are `src/lweight.py`, `log-convexity/symmetry.py` and `log-convexity/linf.py`; the two modes of §4 are `log-convexity/modes.py`; the obstructions and reformulations are `src/lspec.py`, `src/lparity.py` and `log-convexity/involution.py`. The laboratory notes with the history of these experiments are `notes/log-convexity.md` and `notes/calkin-wilf.md`.

Data availability statement

The code and data that support the findings of this note are openly available in the repository `https://github.com/dawidrylko/lattice-dimers` at the tag `king-entropy-v1.0.0`, and are archived on Zenodo, DOI: 10.5281/zenodo.23686293.

Disclosure statement

The author reports no competing interests.

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